Sec.4.1 Increasing and Decreasing Functions

function is increasing on a particular interval if for any, $x_1 < x_2$ then $f(x_1) < f(x_2)$
i.e: As x increases, y increases.

A function is decreasing on a particular interval if for any, $x_1 < x_2$ then $f(x_1) > f(x_2)$.
i.e: As x increases, y decreases.

For a continuous and differentiable function, f(x), the y-values are increasing for all x-values when $f'(x) > 0$ and the y-values are decreasing for all x-values when $f'(x) < 0$.

Eg.1: Find the intervals of increase and decrease for the function, $f(x) = 2x^3 - 3x^2 - 12x$

$$f'(x) = 6x^2 - 6x - 12$$
$$0 = 6x^2 - 6x - 12$$
$$0 = x^2 - x - 2$$
$$= (x-2)(x+1)$$
$$x = 2, -1$$

Interval	Value of f '(x)	Slope of Tangent	Y-values are
$x < -1$	+	POS	increasing
$-1 < x < 2$	—	NEG	decreasing
$x > 2$	+	POS	increasing

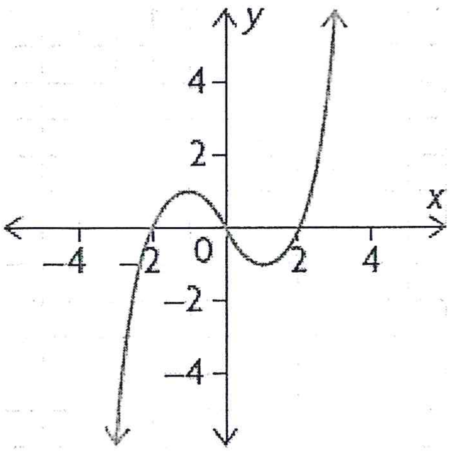
g.2: Find the intervals of increase and decrease for the function, $f(x) = x - 3x^{\frac{1}{3}}$

$$f'(x) = 1 - x^{-2/3}$$
$$0 = 1 - x^{-2/3}$$
$$x^{-2/3} = 1$$
$$x = 1^{-3/2}$$
$$x = \pm 1$$

Interval	Value of f '(x)	Slope of Tangent	Y-values are
$x < -1$	+	POS	↑
$-1 < x < 1$	DNE		
$x > 1$	−	NEG	↓

← what's happening?

Eg.3: Consider the graph of f '(x). Sketch a possible graph of f(x).

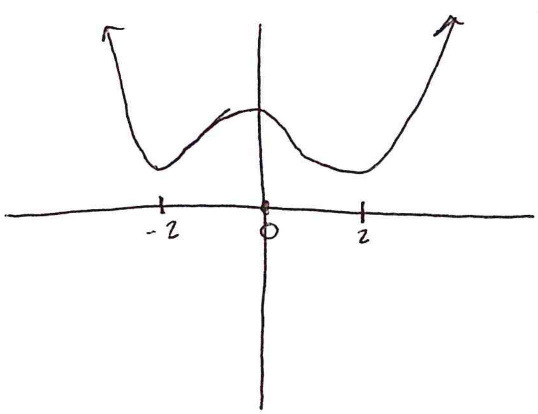


Solution: When the derivative, f '(x) is positive (above the x-axis), the graph of f(x) will be increasing When the derivative is negative (below the x-axis), the graph of f(x) is decreasing The points on f '(x) = 0 are the critical points or turning points on the graph of f(x).

In this example, f '(x) = 0 @ x = $-2, 0, 2$ therefore these are the turning points on the graph of f(x).

Interval	Value of f '(x)	f(x) will be
$x < -2$	−	↓
$-2 < x < 0$	+	↑
$0 < x < 2$	−	↓
$x > 2$	+	↑

Therefore one possible graph of f(x) is;



Sec. 4.2 Critical Points, Local Maxima & Local Minima

ı algorithm for finding the critical points and whether a function has a local maximum or local minimum.

First Derivative Test

1. Find the critical points of the function; that is, determine where f '(x) = 0 or where f '(x) is undefined, for x values in the domain.
2. Use the first derivative test to analyze whether the function, f, is increasing or decreasing on either side of each critical point.
3. Conclude that each critical point locates either a local maximum value of the function, f, or a local minimum value, or neither.

Remember there are three ways in which derivative fail to exist at certain points.

- Cusp
- Vertical Tangent
- Discontinuity

Eg.1: Determine the critical points for $y = x^3 - 6x^2$. Sketch the graph.

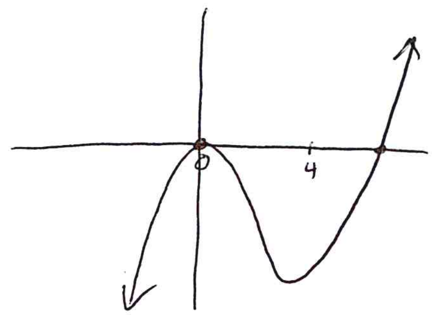
Solution:

$y' = 3x^2 - 12x$

$0 = 3x^2 - 12x$

$0 = 3x(x-4)$

$x = 0, 4$



Interval	Value of f '(x)	f(x) will be
$x < 0$	+	↑
$0 < x < 4$	−	↓
$x > 4$	+	↑

4 will be a local maximum and 0 will be a local minimum.

Eg.2: Determine the critical points for $y = x^4 - 8x^3 + 18x^2$ and state whether these values gives a local maximum, a local minimum or neither for the function.

Solution:

$y' = 4x^3 - 24x^2 + 36x$

$0 = 4x^3 - 24x^2 + 36x$

$= 4x(x^2 - 6x + 9)$

$= 4x(x-3)(x-3)$

$x = 0, 3$

Interval	Value of f '(x)	f(x) will be
$x < 0$	−	↓
$0 < x < 3$	+	↑
$x > 3$	+	↑

local min at $x=0$

neither max nor min at $x=3$

X = 0 will be a local minimum, at the critical point of X = 3 there is neither a local maximum nor a local minimum since the function values increase toward this point and increase away from it as well.

N.B. Critical points that occur when $\frac{dy}{dx} = 0$ give locations of horizontal tangents on the graph of a function. Critical points that occur when $\frac{dy}{dx}$ is undefined give the locations of either vertical tangents or cusps (where we say that no tangent exists).

Read Example # 4 (p.177 of textbook)

Sec. 4.3a Horizontal and Vertical Asymptotes

So far the algorithm for curve sketching is;

- Determine the x & y intercepts (x-intercept you let y = 0 and y-intercept you let x = 0)
- Use the first derivative test to find the critical points and the intervals of increase/decrease
- Identify all local extrema (ie: local maximum and local minimum)
- Determine all asymptotes

Recall: There are 3 kinds of asymptotes: (Vertical, Horizontal and Slant/Oblique)

Vertical Asymptotes

A rational function of the form $f(x) = \frac{p(x)}{q(x)}$ has a vertical asymptote $x = c$ if $q(c) = 0$ and $p(c) \neq 0$.

Essentially we are determining the restrictions on the domain of the function.

The graph of $f(x)$ has a vertical asymptote, $x = c$, if one of the following infinite limit statements is true:

$$\lim_{x \to c^-} f(x) = +\infty, \quad \lim_{x \to c^-} f(x) = -\infty, \quad \lim_{x \to c^+} f(x) = +\infty \text{ or } \lim_{x \to c^+} f(x) = -\infty$$

Eg.1: Find any discontinuities, state the equation of any vertical asymptotes. Describe the behavior of the graph as x approaches the V.A.

a) $y = \frac{x+2}{x-2}$ V.A. @ x = 2

b) $y = \frac{2x}{x^2 - 1}$ V.A. @ x = 1 and x = -1

Horizontal Asymptotes

We need to assess the value of the function as x approaches ± ∞. We use the notation:

$$\lim_{x \to +\infty} f(x) \text{ and } \lim_{x \to -\infty} f(x) \qquad \text{N.B. } \lim_{x \to +\infty} \frac{1}{x} = 0 \text{ and } \lim_{x \to -\infty} \frac{1}{x} = 0$$

The horizontal asymptote for any polynomial can be found by writing it so that the highest degree is a factor.

$p(x) = 3x^2 - 4x + 5$

$p(x) = 3x^2\left(1 - \frac{4}{3x} + \frac{5}{3x^2}\right)$ as $x \to +\infty$ *and* $x \to -\infty$ the bracket approaches 1. Thus the function is essentially p(x) = 3x². As x approaches infinite so does the function therefore no HA.

Eg.2: Determine the equations of any horizontal asymptotes of the function: $f(x) = \frac{3x+5}{2x-1}$. Assess the behavior as $x \to +\infty$ *and* $x \to -\infty$.

Solution:

$$f(x) = \frac{3x\left(1+\frac{5}{3x}\right)}{2x\left(1-\frac{1}{2x}\right)} = \frac{3\left(1+\frac{5}{3x}\right)}{2\left(1-\frac{1}{2x}\right)}$$

$$\frac{3\lim_{x\to\infty}\left(1+\frac{5}{3x}\right)}{2\lim_{x\to\infty}\left(1-\frac{1}{2x}\right)} = \frac{3\left(\lim_{x\to\infty} 1 + \lim_{x\to\infty}\frac{5}{3x}\right)}{2\left(\lim_{x\to\infty} 1 - \lim_{x\to\infty}\frac{1}{2x}\right)} = \frac{3}{2}$$

$$\frac{3\lim_{x\to-\infty}\left(1+\frac{5}{3x}\right)}{2\lim_{x\to-\infty}\left(1-\frac{1}{2x}\right)} = \frac{3}{2}$$

When f is very large. i.e: f(1000) > $3/2$, therefore approaches the asymptote from above.
When f is very small. i.e: f(-1000) < $3/2$, therefore approaches the asymptote from below.

Eg.3: (putting it all together)

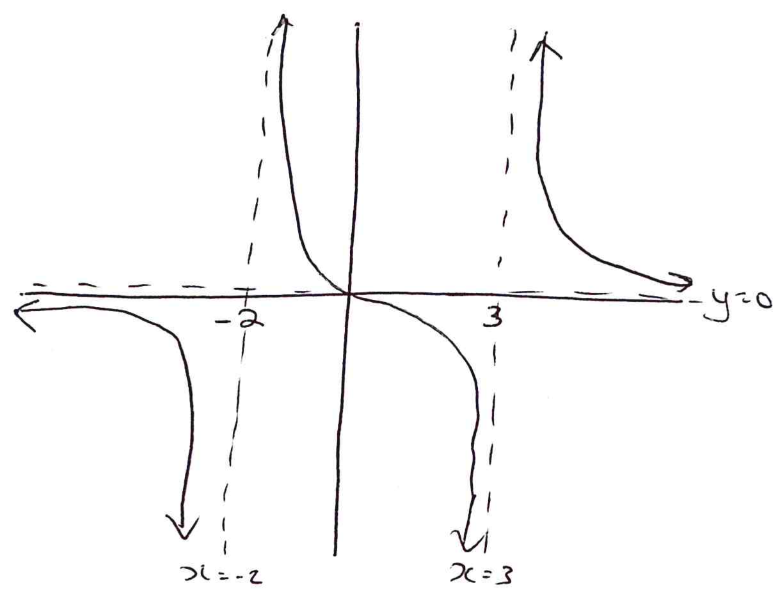
For the function $y = \frac{x}{x^2 - x - 6}$ determine the equations of all V.A. and H.A. Sketch the graph.

$$y = \frac{x}{(x-3)(x+2)}$$

V.A.

$x - 3 = 0$ $\quad$ $x + 2 = 0$

$x = 3$ $\quad$ $x = -2$



H.A.

$$y = \frac{\cancel{x}}{x^{\cancel{2}}\left(\cancel{1} - \frac{1}{x^2} - \frac{6}{x^2}\right)}$$

$$= \frac{1}{x\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)}$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{x}$$

$$= 0$$

check behaviour around V.A.

$y(4) = \frac{4}{(4-3)(4+2)} = \frac{2}{3}$

$y(2) = \frac{2}{(2-3)(2+2)} = -\frac{1}{2}$

$y(-1) = \frac{-1}{(-1-3)(-1+2)} = \frac{1}{4}$

$y(-3) = \frac{-3}{(-3-3)(-3+2)} = -\frac{1}{2}$

Sec. 4.3b Oblique Asymptotes

Oblique asymptotes are straight lines that are not parallel to the axes that functions (curves) approach infinitely closely. They occur in rational functions in which the degree of the numerator polynomial exceeds the degree of the denominator polynomial by exactly 1.

To determine the equation of any O.A. divide the denominator into the numerator using long division.

Eg.1: Determine any asymptotes for the following function.

$$f(x)=\frac{3x^2+10x+5}{3x+4}$$

$$\begin{array}{r} x+2 \\ 3x+4\overline{)3x^2+10x+5} \\ \underline{3x^2+4x} \\ 0x^2+6x+5 \\ \underline{6x+8} \\ -3 \end{array}$$

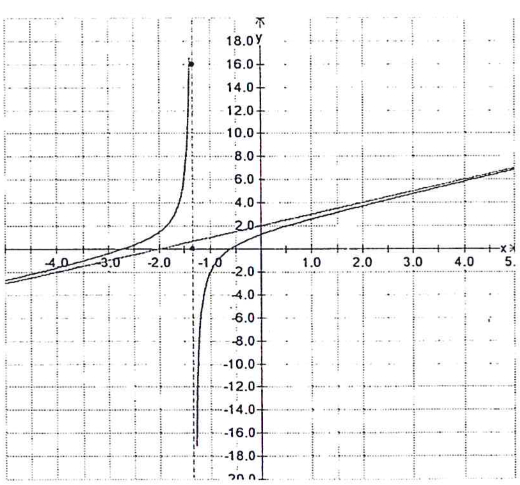
$\therefore f(x)=x+2-\dfrac{3}{3x+4}$ This means that $x+2$ is the O.A.

When x is large and positive the difference between the function and the O.A. is represented by;

$\left(x+2-\dfrac{3}{3x+4}\right)-(x+2)=-\dfrac{3}{3x+4}$. The difference is negative and approaches from below.

When x is large and negative the difference between the function and the O.A. is represented by;

$\left(x+2-\dfrac{3}{3x+4}\right)-(x+2)=-\dfrac{3}{3x+4}$. The difference is positive and approaches from above.

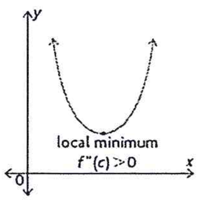


Sec. 4.4 Concavity and Point of Inflection

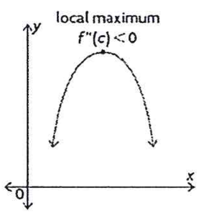
We have seen that the first derivative when set equal to zero will give us the critical points of a function and thus we can find the intervals of increase and decrease. This allows us to identify local maximum and local minimum points.

If y = f(x) has a critical point at x = c, with $f'(c) = 0$, then the behaviour of f(x) at x = c can be analyzed through the use of the 2nd derivative test by analyzing $f''(c)$, as follows:

a. The graph is concave up, and $x = c$ is the location of a local minimum value of the function, if $f''(c) > 0$.

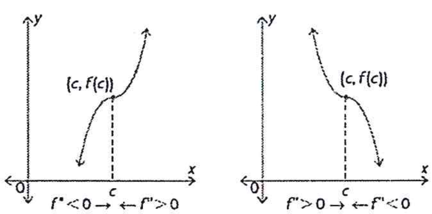


b. The graph is concave down, and $x = c$ is the location of a local maximum value of the function, if $f''(c) < 0$.



c. If $f''(c) = 0$, the nature of the critical point cannot be determined without further work.

A Point of Inflection (POI) occurs at $(c, f(c))$ on the graph of y = f(x) if $f''(x)$ changes sign at x = c. That is, the curve changes from Concave down to concave up, or vice versa.



Eg.1: Find the critical points for the following functions. Determine whether the functions have a local maximum or minimum.

a) $f(x) = x^3 - 6x^2 - 15x + 10$

$f'(x) = 3x^2 - 12x - 15$

$0 = 3(x^2 - 4x - 5)$

$= 3(x-5)(x+1)$

$x = 5, -1$

$f''(x) = 6x - 12$

$f''(5) = 6(5) - 12$
$= 18$
$\therefore$ min

$f''(-1) = 6(-1) - 12$
$= -18$
$\therefore$ max

b) $f(x) = 2x^3 - 10x + 3$

$f'(x) = 6x^2 - 10$

$0 = 6x^2 - 10$

$10 = 6x^2$

$\frac{5}{3} = x^2$

$\pm\sqrt{5/3} = x$

$f''(x) = 12x$

$f''(\sqrt{\tfrac{5}{3}}) = 12\sqrt{\tfrac{5}{3}}$
$=$
$\therefore$ min

$f''(-\sqrt{\tfrac{5}{3}}) = 12(-\sqrt{\tfrac{5}{3}})$
$=$
$\therefore$ max

Eg.2: Discuss the function $y = \dfrac{x}{x^2+1}$ with respect to the concavity and points of inflection.

$y' = \dfrac{(1)(x^2+1) - (2x)(x)}{(x^2+1)^2}$

$= \dfrac{x^2+1-2x^2}{(x^2+1)^2}$

$= \dfrac{-x^2+1}{(x^2+1)^2}$